TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 353, Number 5, Pages 1755–1766
S 0002-9947(01)02638-1
Article electronically published on January 16, 2001

PFAFFIAN SYSTEMS WITH DERIVED LENGTH ONE. THE CLASS OF FLAG SYSTEMS

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ABSTRACT. The incidence relations between a Pfaffian system and the characteristic system of its first derived system lead to obtain a characterization of Pfaffian systems with derived length one. Also, for flag systems, several properties are studied. In particular, an intrinsic proof of a result which determines the class of a system and of all the derived systems is given.

1. INTRODUCTION

In this paper, by means of a structure tensor and the incidence relations between a Pfaffian system S and the characteristic system of the first derived system S_1, characterizations for the integrability of S_1 are obtained. In this paper we introduce new proof techniques which allow us to obtain the results in an instinsic way, not by using coordinates, as was traditional in earlier works concerning Pfaffian systems. So, for example, we give an intrinsic proof of a result that allows us to determine the class of a Pfaffian system and of its first derived system whenever the second derived system there exists and each of them has codimension one in the previous system. This result is particularly interesting for the flag case.

We have attempted to write this paper in a reasonably self-contained form. So, in Section 2, we recall the basic definitions and theorems from the theory of Pfaffian systems we will need.

In Section 3 we present a structure tensor defined by [8]. (We point out that we are not dealing with the well known Sternberg's structure tensor for G-structures.) This tensor $\mathbf{k}$ leads to a reformulation of Frobenius' theorem and it is used to define the characteristic and the derived systems of a Pfaffian system S.

This tensor $\mathbf{k}$, together with the study of the incidence between S and the characteristic system of the first derived system, is used to obtain (Theorem 4.1) a characterization of the integrability of S_1. Also in Section 4, by using the same technique, other properties of Pfaffian systems are obtained.

In Section 5 the main goal is to characterize, in terms of the class, the flag systems (Theorem 5.4 and Corollary 5.3). However, we obtain results which can be applied in a more general situation.

We are interested in emphasizing that the technique used allows us to give direct and intrinsic proofs. Thus, for example, the necessary condition in Theorem 5.4

Received by the editors August 3, 1999 and, in revised form, October 1, 1999.
2000 *Mathematics Subject Classification.* Primary 58A17; Scondary 53C10.
Key words and phrases. Pfaffian systems, structure tensor, characteristic system, derived system, flag systems.
Research partially supported by a DGICYT grant PB94-0796.

was proved by Kumpera-Ruiz [8] and Gardner [5] independently, but the proof we give is not by using coordinates.

Likewise, we think it is interesting to compare the results obtained in Section 5 with some results that appear in [5]. By means of an intrinsic proof we obtain, in Corollary 5.2, the same conclusion from a different hypothesis. Also, we improve the necessary condition in Theorem 5.4 because we do not need to make any assumption about the integrability of S_2.

In Sections 4 and 5 several examples and counterexamples are shown in order to prove that, in most cases, it is not possible to improve the result. We also present an open problem in Section 4.

2. Pfaffian systems

Our purpose in this section is to recall some basic definitions and results from the theory of Pfaffian systems. We refer the reader to [1] and [8] for the proofs unless otherwise indicated.

Let M be a differentiable manifold of finite dimension n which we will assume to be connected and of class C^∞. We will denote by T and T^* the tangent and the cotangent bundle of M respectively. Let $\Gamma(E)$ denote the module of the global sections of a vector bundle E over M, and let $\Gamma_\ell(E)$ be the presheaf of the local sections of E.

Definition 2.1. A Pfaffian system on M is a vector subbundle locally trivial S of T^*.

Thus, if S has rank p, locally S is generated by linearly independent p Pfaffian forms

$$\omega^1, ..., \omega^p \in \Gamma_\ell(S).$$

We shall represent

$$S = span\{\omega^1, ..., \omega^p\}$$

omitting both the point and the neighborhood unless necessary to avoid misunderstandings.

Definition 2.2. Given a Pfaffian system S, an integral manifold of S is a submanifold $\imath : N \hookrightarrow M$ such that $\imath^*(S) = 0$.

Necessarily $dim(N) \leq dim(M) - rank(S)$.

Definition 2.3. A Pfaffian system S of rank p is called completely integrable if, through each point of M, where S is defined, there passes an $(n-p)$-dimensional integral manifold of S.

Definition 2.4. A maximal integral manifold of a Pfaffian system S on M is a connected integral manifold of a maximal dimension of S whose image in M is not a proper subset of any other connected integral manifold.

Theorem 2.1. *Let S be a completely integrable Pfaffian system on M. Then through each point x in M there passes a unique maximal integral manifold of S, and every connected integral manifold of S through x is contained in the maximal one.*

Given a Pfaffian system S, its annihilator $\Sigma = S^{\perp}$ is a subbundle of T, that is, a differentiable distribution. We will identify

$$(T/\Sigma)^* \equiv S \qquad (T^*/S) \equiv \Sigma^*.$$

Moreover, Σ is completely integrable if and only if S is completely integrable and Σ and S have the same integral manifolds.

The integrability of S (or the one of the associated distribution) is characterized by Frobenius' theorem, whose classical version for Pfaffian systems can be stated in the following terms:

Theorem 2.2 (Frobenius)**.** *Let S be a Pfaffian system generated by linearly independent 1–forms $\omega^1, ..., \omega^p$. Then S is completely integrable if and only if $d\omega^i \wedge \omega^1 \wedge ... \wedge \omega^p = 0$, $1 \leq i \leq p$.*

If Frobenius' theorem is satisfied, then, in a neighborhood, there exist local coordinates

$$(x^1, ..., x^n)$$

such that S is locally generated by

$$dx^1, ..., dx^p.$$

This is the standard flat model for the Pfaffian system. A set of functions

$$\{x^1, ..., x^p\}$$

as above, is called a complete set of local first integrals of the completely integrable system S. The maximal integral manifolds, in this neighborhood, are given by

$$\{x^1 = c^1, ..., x^p = c^p\}$$

where $c^1, ..., c^p$ are real constants depending on the point.

Definition 2.5. An automorphism or a symmetry of S (also of Σ) is a diffeomorphism φ of M such that $\varphi^*(S) = S$ (or $\varphi_*(\Sigma) = \Sigma$). We shall denote $\mathcal{A}ut(S)$ the group of automorphisms of S.

Definition 2.6. An infinitesimal automorphism, or an infinitesimal symmetry, of S (also of Σ) is a vector field which generates a local 1–parameter group of automorphisms of S. We will denote $L(S)$ the set of all the infinitesimal automorphisms of S.

It is easily checked that $\xi \in L(S)$ if and only if

$$L_\xi(\Gamma_\ell(S)) \subset \Gamma_\ell(S) \quad (\text{or } L_\xi(\Gamma_\ell(\Sigma)) \subset \Gamma_\ell(\Sigma))$$

where L_ξ is the Lie derivative with respect to ξ.

Definition 2.7. An infinitesimal automorphism $\xi \in L(S)$ is called characteristic if it is tangent to Σ, that is, if $\xi \in \Gamma_\ell(\Sigma)$.

The module

$$L_c(S) = L(S) \cap \Gamma_\ell(\Sigma)$$

of the characteristic vector fields induces a distribution, which might be singular, and which we shall denote by $\Delta = \Delta(S)$: in each point x of M, Δ_x is the subspace of T_xM generated by the values in x of the characteristic vector fields.

Since $\varphi \in L_c(S)$ if and only if

$$i_\varphi \omega = i_\varphi i_\eta d\omega = 0$$

for every $\omega \in \Gamma_\ell(S)$ and every $\eta \in \Gamma_\ell(\Sigma)$, the set of the vector fields of M which take value in Δ is $L_c(S)$.

Whenever the dimension of Δ_x is constant, that is, whenever Δ is a distribution, it is called, following Élie Cartan, the characteristic distribution of S and its annihilator $C = C(S) \subset T^*$ is the characteristic system of S. In this case, the integer $rank(C(S))$ is named the class of the Pfaffian system S and it is denoted by $class(S)$.

Henceforth, we shall assume that S is a Pfaffian system with constant class, that is, $dim(\Delta_x)$ is constant.

Theorem 2.3. *The characteristic system C of a Pfaffian system S is always a completely integrable system.*

Proposition 2.1. *Let S be a Pfaffian system. The following are equivalent:*

(a) *S is completely integrable.*
(b) $\Delta = \Sigma$.
(c) $C = S$.

Theorem 2.4. *Let S be a Pfaffian system. Then the characteristic system of S is the smallest completely integrable Pfaffian system with the property that if $\{x^1, ..., x^p\}$ is a local system of first integrals, then locally there exist generators of S which only depend on $\{x^1, ..., x^p\}$ and their differentials.*

The class, therefore, is the minimum number of variables necessary in order to write down local generators of the system. Those variables are called the characteristic variables of the system S.

3. Structure tensor. Derived systems

The structure tensor of a Pfaffian system S ([8]) is defined as the vector bundle morphism

$$\mathbf{k} : \Sigma \otimes S \longrightarrow \Sigma^* \equiv T^*/S$$

given by

$$\mathbf{k}(\xi \otimes \omega) = (\mathbf{q} \circ L_\xi)(\omega)$$

for all $\xi \in \Gamma(\Sigma)$ and $\omega \in \Gamma(S)$, where

$$\mathbf{q} : T^* \longrightarrow T^*/S$$

is the quotient morphism.

Notice that $\quad \mathbf{k}(\xi \otimes \omega) = \mathbf{q}(i_\xi(d\omega))$.

Proposition 3.1. *The rank of $\mathbf{k}$ is constant if and only if the dimension of Δ_x is constant.*

In this case, we establish:

Theorem 3.1. *Let S be a Pfaffian system of constant class, then*

$$C(S) = \boldsymbol{q}^{-1}(\mathbf{k}(\Sigma \otimes S))$$

and

$$class(S) = rank(\mathbf{k}) + rank(S)$$

We can formulate Frobenius' theorem by means of the structure tensor in the following terms:

Theorem 3.2. *A Pfaffian system S is completely integrable if and only if* $\mathbf{k} \equiv 0$.

Remark 3.1. Theorem 3.2 allows us to consider the structure tensor of a Pfaffian system as a curvature for the flatness (integrability) of a Pfaffian system.

Proposition 3.2. *The rank of* $\mathbf{k}$ *is always different from* 1*; as a consequence, the class of a Pfaffian system S is either equal to its rank (if S is completely integrable) or strictly greater than* rank$(S) + 1$ *(if S is not completely integrable).*

Definition 3.1. In each point $x \in M$, $(S_1)_x$ is defined as the right kernel of the tensor $\mathbf{k}_x$, i.e.,

$$(S_1)_x = \{\omega \in S_x \,/\, \mathbf{k}(\xi \otimes \omega) = 0,\, \forall \xi \in \Sigma_x\}.$$

Whenever the dimension of these spaces is constant, the Pfaffian system S_1 obtained is called, following Élie Cartan, the first derived system of S.

Notice that, since

$$d\omega(\xi, \eta) = -\omega([\xi, \eta]),\ \forall \omega \in \Gamma_\ell(S),\ \xi, \eta \in \Gamma_\ell(\Sigma)$$

the field $\Sigma_1 = S_1^\perp$, consisting of the annihilators of S_1, is the distribution induced by the submodule of $\Gamma_\ell(T)$ generated by

$$\Gamma_\ell(\Sigma) \cup [\Gamma_\ell(\Sigma), \Gamma_\ell(\Sigma)].$$

Theorem 3.3. *A Pfaffian system S is completely integrable if and only if $S = S_1$.*

Since S_1 is a subbundle of S canonically associated to S, we have

Theorem 3.4.

$$\mathcal{A}ut(S) \subset \mathcal{A}ut(S_1), \qquad L(S) \subset L(S_1),$$

$$L_c(S) \subset L_c(S_1), \qquad \Delta(S) \subset \Delta(S_1).$$

We shall denote $\Delta(S_1) = \Delta_1$, $C(S_1) = C_1$ and $\mathbf{k}_1$ the structure tensor of S_1.
The successive derived systems are defined inductively by

$$S_{\mathrm{r}+1} = (S_{\mathrm{r}})_1$$

Whenever all the derived systems exist we obtain a decreasing sequence of Pfaffian systems,

$$S = S_0 \supseteq S_1 \supseteq \ \dots\ \supseteq S_{\mathrm{r}} \supseteq S_{\mathrm{r}+1} \supseteq \ \dots$$

whose last term is a completely integrable Pfaffian system (which might be trivial).

Definition 3.2. A Pfaffian system which satisfies the above condition is called totally regular. The derived length of a totally regular Pfaffian system S is the smallest nonnegative integer ℓ such that $S_\ell = S_{\ell+1}$.

For the r^{th} derived system, S_{r} of S, $\mathbf{k}_{\mathrm{r}}$ represents its structure tensor and we denote

$$\Delta_{\mathrm{r}} = \Delta(S_{\mathrm{r}}),\, C_{\mathrm{r}} = C(S_{\mathrm{r}}).$$

Theorem 3.5.

$$\mathbf{k}_1(\Sigma \otimes S_1) \subseteq S/S_1$$

Proof. Since the diagram

$$\begin{tikzcd}
\Sigma \otimes S \arrow[r, "\mathbf{k}"] & T^*/S \\
\Sigma \otimes S_1 \arrow[u, "\imath"] \arrow[r, "\mathbf{k}_1"'] & T^*/S_1 \arrow[u, "\pi"']
\end{tikzcd}$$

is commutative and the sequence

$$0 \longrightarrow S/S_1 \longrightarrow T^*/S_1 \longrightarrow T^*/S \longrightarrow 0$$

is exact, condition

$$\mathbf{k}(\Sigma \otimes S_1) = \{0\}$$

implies

$$\mathbf{k}_1(\Sigma \otimes S_1) \subseteq S/S_1.$$

□

This property, which is essential in this work, allows us, for example, to define, for a totally regular Pfaffian system with derived length ℓ, the reduced tensor by

$$\mathbf{k}_{(\mathrm{r})} : \Sigma \otimes S_{\mathrm{r}} \longrightarrow S_{\mathrm{r}-1}/S_{\mathrm{r}}, \quad 1 \leq \mathrm{r} \leq \ell,$$

as the restriction of the structure tensor

$$\mathbf{k}_{\mathrm{r}} : \Sigma_{\mathrm{r}} \otimes S_{\mathrm{r}} \longrightarrow T^*/S_{\mathrm{r}}$$

and $\mathbf{k}_{(0)} = \mathbf{k}$.

Since S_{N} ($\mathrm{N} \geq \ell$) is completely integrable, $\mathbf{k}_{\mathrm{N}} = \mathbf{k}_{(\mathrm{N})} = 0$ for every integer $\mathrm{N} \geq \ell$.

4. The characteristic system of the first derived system

In this section the goal is to prove some results concerning the incidence relations between a Pfaffian system S and the characteristic system of its first derived system S_1. These relations allow us to characterize the integrability of S_1, that is, we characterize the totally regular Pfaffian systems with derived length one.

Theorem 4.1. *Let S be a totally regular Pfaffian system with derived length ℓ. The following conditions are equivalent:*

(a) $\ell = 1$, *that is,* S_1 *is completely integrable.*
(b) $C_1 \cap S = S_1$.
(c) $\mathbf{k}_1(\Sigma \otimes S_1) = \{0\}$.
(d) $C_1 \cap S = C_1$.
(e) $\mathbf{k}_1(\Sigma \otimes S_1) = C_1/S_1$.
(f) *For every complement* $\widehat{\Sigma}$ *of* Σ *in* Σ_1, $\mathbf{k}_1(\widehat{\Sigma} \otimes S_1) = \{0\}$.
(g) *There exists a complement* $\widehat{\Sigma}$ *of* Σ *in* Σ_1 *such that* $\mathbf{k}_1(\widehat{\Sigma} \otimes S_1) = \{0\}$.

Proof. S_1 is completely integrable iff $\mathbf{k}_1 \equiv 0$ iff $C_1 = S_1$. Therefore, if we suppose that S_1 is completely integrable, we have

$$C_1 \cap S_1 = S_1 \cap S = S_1 = C_1, \quad \mathbf{k}_1 \equiv 0, \quad C_1/S_1 = \{0\}$$

and, as a consequence, conditions (b) through (g) are necessary for S_1 to be completely integrable.

Let us suppose $C_1 \cap S = S_1$. Since

$$\mathbf{k}_1(\Sigma \otimes S_1) \subseteq S/S_1 \quad \text{and} \quad \mathbf{k}_1(\Sigma_1 \otimes S_1) = C_1/S_1,$$

it follows that

$$\mathbf{k}_1(\Sigma \otimes S_1) \subseteq (C_1 \cap S)/S_1 = \{0\}$$

that is, we get (c).

If $\mathbf{k}_1(\Sigma \otimes S_1) = \{0\}$ then the local sections of Σ are infinitesimal automorphisms of S_1, i.e., $\Sigma \subseteq \Delta_1$, or, equivalently, $C_1 \subseteq S$ and (d) holds.

Moreover, C_1 is a proper subset of S if S is not completely integrable because C_1 is always a completely integrable system.

From (d), as $C_1 \cap S = C_1$ is equivalent to $C_1 \subseteq S$, we have

$$S_1 \subseteq C_1 \subseteq S.$$

Since C_1 is completely integrable and $\Sigma \subseteq \Delta_1$, we obtain

$$\mathbf{k}(\Sigma \otimes C_1) = \{0\}.$$

This means that C_1 is included in the right kernel of $\mathbf{k}$, which is S_1. Therefore, $C_1 = S_1$ is completely integrable, and so the first four conditions are equivalent.

Condition (e) is also equivalent to the ones above because, if

$$\mathbf{k}_1(\Sigma \otimes S_1) = C_1/S_1,$$

then $C_1 \subseteq S$, i.e., $C_1 \cap S = C_1$ and thus (e) implies (d).

Finally, from (g) if, for a complement $\widehat{\Sigma}$ of Σ in Σ_1,

$$\mathbf{k}_1(\widehat{\Sigma} \otimes S_1) = \{0\},$$

then

$$\mathbf{k}_1(\Sigma \otimes S_1) = C_1/S_1,$$

that is, S_1 is completely integrable. □

In terms of the reduced tensor $\mathbf{k}_{(1)}$ we can state conditions (c) and (e) in the following corollary:

Corollary 4.1. *The following are equivalent:*

i) S_1 *is completely integrable (i.e.,* $\mathbf{k}_1 \equiv 0$*).*

ii) $\mathbf{k}_{(1)} \equiv 0$.

iii) $Im(\mathbf{k}_1) = Im(\mathbf{k}_{(1)})$ *(i.e.,* $\mathbf{k}_1(\Sigma_1 \otimes S_1) = \mathbf{k}_{(1)}(\Sigma \otimes S_1)$*).*

Proposition 4.1. *If* $rank(\mathbf{k}_{(1)}) = dim(S/S_1) = p$*, then* $S \subseteq C_1$*.*

Moreover, $S = C_1$ *if and only if* $p = 0$*.*

Proof. Since $\mathbf{k}_1(\Sigma \otimes S_1) \subseteq S/S_1$, we have $\mathbf{k}_1(\Sigma \otimes S_1) = S/S_1$ and then $S \subseteq C_1$. Moreover, $p = 0$ iff S is completely integrable iff $S = S_1 = C_1$. □

As a consequence of Theorem 4.1 and Proposition 4.1 we have a new corollary:

Corollary 4.2. *If* $dim(S/S_1) = 1$*, then or* S_1 *is completely integrable or* $S \subsetneq C_1$*.*

Proof. As $0 \leq rank(\mathbf{k}_{(1)}) = dim(\mathbf{k}_1(\Sigma \otimes S_1)) \leq dim(S/S_1) = 1$, then

i) $dim(\mathbf{k}_1(\Sigma \otimes S_1)) = 0 \Longleftrightarrow S_1$ is completely integrable,

or

ii) $dim(\mathbf{k}_1(\Sigma \otimes S_1)) = 1 = dim(S/S_1) \Longrightarrow S \subsetneq C_1$. □

Remark 4.1. Example 4.1 shows that if the codimension of S_1 in S is greater than one, the nonintegrability of S_1 does not imply $S \subset C_1$.

Example 4.1. In dimension 6, the Pfaffian system S of rank 3, locally generated by the 1-forms

$$\begin{aligned} \omega^1 &= dx^1 + x^2 dx^6 \\ \omega^2 &= dx^2 + x^3 dx^6 \\ \omega^3 &= dx^4 + x^5 dx^6 \end{aligned}$$

has as its first derived system S_1, the system of constant rank 1,

$$S_1 = span\{\omega^1\}.$$

In this case $\omega^3 \notin C_1 = span\{dx^1, dx^2, dx^6\}$ and thus $S \not\subset C_1$.

Proposition 4.2. *If any of the next conditions hold:*

(a) *S_1 is completely integrable.*
(b) *$dim(S/S_1) = 1$.*

Then $Im(\mathbf{k}_{(1)}) = (C_1 \cap S)/S_1$ *(i.e.,* $\mathbf{k}_1(\Sigma \otimes S_1) = (C_1 \cap S)/S_1$*).*

Proof. For (a), it is enough to take into account that $\mathbf{k}_1 = 0$ and $C_1 \cap S = S_1$.
For (b) as $\mathbf{k}_1(\Sigma \otimes S_1) \subseteq S/S_1$ and $dim(S/S_1) = 1$
or $\mathbf{k}_1(\Sigma \otimes S_1) = \{0\}$; that is, S_1 is completely integrable,
or $dim\, \mathbf{k}_1(\Sigma \otimes S_1) = 1$, i.e., $\mathbf{k}_1(\Sigma \otimes S_1) = S/S_1$ and then $S/S_1 \subset C_1/S_1$.

Thus $S/S_1 = (C_1 \cap S)/S_1$, i.e., $\mathbf{k}_1(\Sigma \otimes S_1) = (C_1 \cap S)/S_1$. □

Remark 4.2. Note that we have proved $\mathbf{k}_1(\Sigma \otimes S_1) = (C_1 \cap S)/S_1$ unless $dim(S/S_1)$ > 1. In this case the problem is still open.

5. Flag Pfaffian systems

In this section the first step is to analyze those Pfaffian systems such that the codimension of its first derived system, S_1, in S is one. Next we study the case $dim(S/S_1) = dim(S_1/S_2) = 1$, which means that the second derived system verifies the same property in relation to the first one. Finally, we extend the results to totally regular systems with derived length ℓ,

$$S_\ell \subset ... \subset S_1 \subset S_0 = S$$

such that, $dim(S_{\mathrm{r}-1}/S_{\mathrm{r}}) = 1$, for $1 \leq \mathrm{r} \leq \ell$, or equivalently,

$$dim(S/S_\ell) = \ell = rank(S) - rank(S_\ell).$$

Also, we obtain a sufficient condition for S_1 to have codimension one in S.

Definition 5.1. A totally regular Pfaffian system S is named a flag (Pfaffian) system when its derived length ℓ is equal to $rank(S)$.

These systems are called *special systems* by Goursat [5]. The most representative example is the contact system in the space of holonomic jets for functions of one variable.

As a first result we can apply Proposition 4.2 to flag systems:

Proposition 5.1. *If S is a flag Pfaffian system with derived length ℓ, then*

$$\mathbf{k}_{\mathrm{i}}(\Sigma_{\mathrm{i}-1} \otimes S_{\mathrm{i}}) = (C_{\mathrm{i}} \cap S_{\,\mathrm{i}-1})/S_{\mathrm{i}}$$

for every $1 \leq \mathrm{i} \leq \ell$.

Theorem 5.1. *Let S be a Pfaffian system with first derived system S_1. If $S \subset C_1$ and $rank(\mathbf{k}_1) = 2$, then $rank(\mathbf{k}) = 2$.*

Proof. As $rank(\mathbf{k}_1) = 2$, S_1 has codimension two in C_1. Since $S_1 \subset S \subset C_1$, we deduce that S has codimension one in C_1 or, equivalently, Δ_1 has codimension one in Σ. So, if $\widehat{\Delta}_1$ is a complement of Δ_1 in Σ, that is, $\Sigma = \Delta_1 \oplus \widehat{\Delta}_1$, then $dim(\widehat{\Delta}_1) = 1$.

On the other hand, from

$$dim(C_1/S_1) = 2 \text{ and } S_1 \subset S \subset C_1$$

it follows that $dim(S/S_1) = 1$ and thus, if $S = S_1 \oplus \widehat{S}_1$, $\widehat{S}_1$ has dimension one.

Therefore we can descompose

$$\Sigma \otimes \widehat{S}_1 = (\Delta_1 \otimes \widehat{S}_1) \oplus (\widehat{\Delta}_1 \otimes \widehat{S}_1)$$

and as $dim(\widehat{\Delta}_1 \otimes \widehat{S}_1) = 1$, we have

$$dim(\mathbf{k}(\widehat{\Delta}_1 \otimes \widehat{S}_1)) \leq 1. \tag{1}$$

Also, since C_1 is completely integrable, the vector fields of Δ_1 are characteristic vector fields of C_1, that is $L_{\Delta_1}(C_1) \subseteq C_1$ and therefore

$$\mathbf{k}(\Delta_1 \otimes \widehat{S}_1) \subseteq C_1/S.$$

This implies

$$dim(\mathbf{k}(\Delta_1 \otimes \widehat{S}_1)) \leq 1. \tag{2}$$

From (1) and (2),

$$rank(\mathbf{k}) \leq 2.$$

But S is not a completely integrable system and so

$$rank(\mathbf{k}) \geq 2$$

and therefore

$$rank(\mathbf{k}) = 2.$$

□

Notice that Gardner [5] obtains the same conclusion, that is, $rank(\mathbf{k}) = 2$ by changing the hypothesis $S \subset C_1$ and $rank(\mathbf{k}_1) = 2$ by $dim(S/S_1) = 1$ and $rank(\mathbf{k}_1) \neq 0$.

We can state the above theorem in terms of the class in the next corollary:

Corollary 5.1. *If $S \subset C_1$ and $class(S_1) = rank(S_1) + 2$, then*

$$class(S) = rank(S) + 2.$$

From the proof of Theorem 5.1 we obtain

Corollary 5.2. *If $rank(\mathbf{k}_1) = 2$ and $S \subset C_1$, then the codimension of S_1 in S is one.*

Remark 5.1. Example 4.1 shows that we cannot deduce $rank(\mathbf{k}) = 2$ from the condition $rank(\mathbf{k}_1) = 2$.

Remark 5.2. Below we give two examples which prove that, even in the flag case, if $rank(\mathbf{k}_1) \neq 2$, in general, the rank of $\mathbf{k}$ cannot be determined.

Example 5.1. Let us consider, in dimension six, the Pfaffian system of rank two, locally generated by

$$\begin{array}{rcl}\omega^1 & = & dx^1 \\ \omega^2 & = & dx^2 + x^3 dx^4 + x^5 dx^6.\end{array}$$

The derived system is a completely integrable system of rank one locally generated by ω^1. So $rank(\mathbf{k}_1) = 0$ whereas $rank(\mathbf{k}) = 4$, that is, the class of S is six, the characteristic system of S in every point is the whole cotangent space in the point.

Example 5.2. Let us consider, in dimension five, the Pfaffian system of rank three, locally generated by the 1-forms:

$$\begin{array}{rcl}\omega^1 & = & dx^1 + (x^3 + x^4 x^5) dx^4 \\ \omega^2 & = & dx^2 + x^3 dx^5 \\ \omega^3 & = & dx^3 + x^4 dx^5.\end{array}$$

The derived system S_1 is generated by ω^1 and ω^2. This system determines completely, in dimension five, the system S, that is, given the system S_1, of rank two locally generated by ω^1 and ω^2, S is the only Pfaffian system on this manifold which has S_1 as its first derived system.

The derived system of S_1, S_2, is trivial in this case and we have

$$rank(\mathbf{k}) = 2 \text{ and } rank(\mathbf{k}_1) = 3;$$

that is, in both cases, the characteristic system of S in each point is the whole cotangent space in the point.

This situation is the one that É. Cartan [4] describes saying that there exists a univocal correspondence between S and S_1 in dimension five (also see [3]).

Theorem 5.2. *If $rank(\mathbf{k}) = 2$, then S_1 has codimension one in S.*

Proof. Since $rank(\mathbf{k}) = 2$, S has codimension two in C, or, equivalently, Δ has codimension two in Σ. Let X_1 and X_2 be linearly independent local generators of a complement of Δ in Σ. Let us call

$$\widetilde{\Sigma} = \Sigma \oplus span\{[X_1, X_2]\}$$

and let $\widetilde{S} = \widetilde{\Sigma}^{\perp}$. As $\widetilde{S} \subset S$ and $dim(S/\widetilde{S}) = 1$ it suffices to prove $\widetilde{S} = S_1$.

Let $\omega \in \Gamma_\ell(\widetilde{S})$. For i,j $= 1, 2$

$$(L_{X_\mathrm{i}}\omega)(X_\mathrm{j}) = d\omega(X_\mathrm{i}, X_\mathrm{j}) = -\omega([X_\mathrm{i}, X_\mathrm{j}]) = 0.$$

For $\xi \in \Gamma_\ell(\Delta)$,

$$(L_{X_\mathrm{i}}\omega)(\xi) = d\omega(X_\mathrm{i}, \xi) = -d\omega(\xi, X_\mathrm{i}) = -(L_\xi\omega)(X_\mathrm{i}) = 0.$$

In addition to this, $\mathbf{k}(\Delta \otimes S) = \{0\}$. Thus $\mathbf{k}(\Delta \otimes \widetilde{S}) = \{0\}$ and therefore $\widetilde{S} \subseteq S_1$. Since $S \neq S_1$, this implies $\widetilde{S} = S_1$. ☐

Theorem 5.3. *If $dim(S/S_1) = 1$ and $dim(S_1/S_2) = 1$, then $rank(\mathbf{k}_1) = 2$.*

Therefore, the class of S_1 is constant and equal to $rank(S_1) + 2$.

Proof. As $dim(S/S_1) = 1$, $\Sigma_1 = \widehat{\Sigma} \oplus \Sigma$ with $dim(\widehat{\Sigma}) = 1$ and

$$\mathbf{k}_1(\Sigma_1 \otimes S_1) = \mathbf{k}_1(\widehat{\Sigma} \otimes S_1) + \mathbf{k}_1(\Sigma \otimes S_1). \tag{3}$$

Since $\mathbf{k}_1(\Sigma \otimes S_1) \subseteq S/S_1$, we have

$$dim(\mathbf{k}_1(\Sigma \otimes S_1)) \leq 1. \tag{4}$$

On the other hand, as S_2 is the first derived system of S_1,

$$\mathbf{k}_1(\Sigma_1 \otimes S_2) = \{0\}$$

which implies, if $\widehat{S}_2$ is a complement of S_2 in S_1,

$$\mathbf{k}_1(\widehat{\Sigma} \otimes S_1) = \mathbf{k}_1(\widehat{\Sigma} \otimes \widehat{S}_2).$$

So, since $dim(\widehat{\Sigma}) = 1$ and $dim(\widehat{S}_2) = 1$, we deduce

$$dim(\mathbf{k}(\widehat{\Sigma} \otimes S_1)) \leq 1. \tag{5}$$

As S_1 is not completely integrable, by Proposition 3.2 we have

$$rank(\mathbf{k}_1) \geq 2. \tag{6}$$

By using from (3) through (6) it follows that

$$rank(\mathbf{k}_1) = 2.$$

□

Remark 5.3. Example 5.2 shows that it is essential $dim(S_1/S_2) = 1$ to obtain $rank(\mathbf{k}_1) = 2$.

The above theorem together with Theorem 5.1 give a direct and intrinsic proof of the next result:

Theorem 5.4.

$$dim(S/S_1) = dim(S_1/S_2) = 1 \textit{ if and only if } rank(\mathbf{k}) = rank(\mathbf{k}_1) = 2.$$

Thus, in this case, S and S_1 have constant class equal to $rank(S) + 2$ and $rank(S_1) + 2$ respectively.

A demonstration using coordinates of the necessary condition of Theorem 5.4 can be seen in [8].

It is possible to reformulate Theorem 5.4 in the following terms:

Theorem 5.5. *Let S be a totally regular Pfaffian system with derived length $\ell \geq 2$:*

$$dim(S/S_2) = 2 \textit{ if and only if } dim(C/S) = dim(C_1/S_1) = 2.$$

Notice that we improve the result obtained by R.B. Gardner in [5], in the sense that we do not need to make any assumption about the integrability of S_2. Aside from this, the proof we give is not in coordinates.

Corollary 5.3. *If S is a flag system, with derived length $\ell \geq 2$, then all the derived systems S_{r}, $0 \leq \mathrm{r} \leq \ell$, have constant class. The class of S_ℓ is equal to its rank and the one of S_{r}, $\mathrm{r} < \ell$, is equal to $rank(S_{\mathrm{r}}) + 2$.*

Reciprocally, if S is a totally regular Pfaffian system with derived length $\ell \geq 2$, such that $rank(\mathbf{k}_{\mathrm{r}}) = 2$, for every $\mathrm{r} < \ell$, then S is a flag system.

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